## Solution of HW3

## **Compulsory Part:**

18 (a) **Proof.** For two nonnegative integers x and y, we have

$$P^{y+1}(x,y) > P(x,0)P(0,1)P(1,2)\cdots P(y-1,y) = (1-p)p^y > 0.$$

By Q16, x leads to y. Hence the chain is irreducible.

(b) Solution. For n = 1,  $P_0(T_0 = 1) = P(0, 0) = 1 - p$ . For  $n \ge 2$ ,  $P_0(T_0 = n) = P(0, 1)P(1, 2) \cdots P(n - 2, n - 1)P(n - 1, 0) = p^{n-1}(1 - p)$ .

(c) **Proof.** Note that  $\rho_{00} = \sum_{n=1}^{\infty} P_0(T_0 = n) = \sum_{n=1}^{\infty} p^{n-1}(1-p) = 1$ . This implies that 0 is recurrent. Since the chain is irreducible, it is recurrent.

20(b) (b) Clearly  $\rho_{\{0,1\}}(0) = \rho_{\{0,1\}}(1) = 1$  and  $\rho_{\{0,1\}}(2) = \rho_{\{0,1\}}(4) = 0$ . By one-step argument, we have

$$\begin{cases} \rho_{\{0,1\}}(3) = 1/2 + (1/4)\rho_{\{0,1\}}(5), \\ \rho_{\{0,1\}}(5) = 1/5 + (1/5)\rho_{\{0,1\}}(3) + (2/5)\rho_{\{0,1\}}(5). \end{cases}$$

Hence  $\rho_{\{0,1\}}(3) = 7/11$  and  $\rho_{\{0,1\}}(5) = 6/11$ .

26 Using (59) (on textbook, page 31), we have

$$P_x(T_0 < T_n) = \frac{\sum_{y=x}^{n-1} \gamma_y}{\sum_{y=0}^{n-1} \gamma_y} = 1 - \frac{\sum_{y=0}^{x-1} \gamma_y}{\sum_{y=0}^{n-1} \gamma_y}$$

for 0 < x < n. Note that for x > 0,  $1 \le T_{x+1} < T_{x+2} < \cdots$ . Hence  $\{T_0 < T_n\}_{n=1}^{\infty}$  forms a nondecreasing sequence of events. By continuity of the probability, we have for  $x \ge 1$ ,

$$\rho_{x0} = P_x(T_0 < \infty) = P_x\left(\bigcup_{n=1}^{\infty} \{T_0 < T_n\}\right) = \lim_{n \to \infty} P_x(T_0 < T_n) = 1 - \lim_{n \to \infty} \frac{\sum_{y=0}^{x-1} \gamma_y}{\sum_{y=0}^{n-1} \gamma_y}$$

(a) If ∑<sub>y=0</sub><sup>∞</sup> γ<sub>y</sub> = ∞, then the above limit is 0 and ρ<sub>x0</sub> = 1.
(b) If ∑<sub>y=0</sub><sup>∞</sup> γ<sub>y</sub> < ∞, then</li>

$$\rho_{x0} = 1 - \frac{\sum_{y=0}^{x-1} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y} = \frac{\sum_{y=x}^{\infty} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y}.$$

27 (a) If  $q \ge p$ , then

$$\sum_{y=0}^{\infty} \gamma_y = \sum_{y=0}^{\infty} \left(\frac{q}{p}\right)^y \ge \sum_{y=0}^{\infty} 1^y = \infty.$$

Hence by Q26(a),  $\rho_{x0} = 1$ .

(b) If q < p, then

$$\sum_{y=0}^{\infty} \gamma_y = \sum_{y=0}^{\infty} \left(\frac{q}{p}\right)^y = \frac{1}{1-\frac{q}{p}} = \frac{p}{p-q} < \infty.$$

Hence by Q26(b) and  $\sum_{y=x}^{\infty} \gamma_y = (q/p)^x \cdot p/(p-q),$  $(q/p)^x \cdot p/(p-q)$ 

$$\rho_{x0} = \frac{(q/p)^{*} \cdot p/(p-q)}{p/(p-q)} = (q/p)^{x}.$$

29 (a) **Proof.** Note that for  $y \ge 1$ ,

$$\gamma_y = \prod_{x=1}^y \frac{q_x}{p_x} = \frac{1^2 \cdot 2^2 \cdots y^2}{2^2 \cdots y^2 \cdot (y+1)^2} = \frac{1}{(y+1)^2}.$$

Therefore,  $\sum_{y=0}^{\infty} \gamma_y = 1 + \sum_{y=1}^{\infty} \frac{1}{(y+1)^2} = \frac{\pi^2}{6} < \infty$ . Hence the chain is transient.

**(b) Solution.** By Q26(b),

$$\rho_{x0} = \frac{\sum_{y=x}^{\infty} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y} = 1 - \frac{6}{\pi^2} \sum_{y=0}^{x-1} \frac{1}{(y+1)^2}.$$

32 Solution. Note that in Example 14, the probability that the male line of a given man eventually becomes extinct is  $\rho = \sqrt{5} - 2$ . Hence if  $X_1 = 2$ , the probability that the male line will continue forever is

$$1 - \rho^2 = 4(\sqrt{5} - 2) \approx 0.9443.$$

34 The mean number of offspring is

$$\mu = \sum_{x=0}^{\infty} x f(x) = \sum_{x=0}^{\infty} p x (1-p)^x = \frac{1-p}{p}.$$

If  $p \ge 1/2$ , then  $\mu \le 1$  and so  $\rho = 1$ .

If p < 1/2, then  $\mu > 1$ . We need to solve

$$t = \sum_{y=0}^{\infty} p(1-p)^y t^y = \frac{p}{1-(1-p)t}$$

or equivalently,

$$(1-p)t^2 - t + p = 0.$$

This equation has two roots 1 and  $\frac{p}{1-p}$ . Consequently,  $\rho = \frac{p}{1-p}$ .

## **Optional Part:**

23 Since 
$$\binom{2d}{y} = \frac{2d}{2d-y} \binom{2d-1}{y}$$
, we have  

$$\sum_{y=0}^{2d} P(x,y) \frac{2d-y}{2d} = \sum_{y=0}^{2d} \binom{2d-1}{y} \left(\frac{x}{2d}\right)^y \left(1 - \frac{x}{2d}\right)^{2d-y}$$

$$= \frac{2d-x}{2d} \sum_{y=0}^{2d-1} \binom{2d-1}{y} \left(\frac{x}{2d}\right)^y \left(1 - \frac{x}{2d}\right)^{2d-1-y}$$

$$= \frac{2d-x}{2d}.$$

Compare with the one-step formula

$$\rho_{\{0\}}(x) = \sum_{y=0}^{2d} P(x,y)\rho_{\{0\}}(y).$$

Hence  $\rho_{\{0\}}(x) = \frac{2d-x}{2d}, \ 0 < x < 2d.$ 

25 Solution. (a) In Q24, let p = 9/19, q = 10/19, d = 1001 and x = 1000. Then

$$P_{1000}(T_0 < T_{1001}) = \frac{\left(\frac{10}{9}\right)^{1001} - \left(\frac{10}{9}\right)^{1000}}{\left(\frac{10}{9}\right)^{1001} - 1} \approx 0.1.$$

(b) The expected loss is

$$1000 \cdot P_{1000}(T_0 < T_{1001}) - P_{1000}(T_0 > T_{1001}) \approx 100 - 0.9 = 99.1.$$

28 If  $p_x \leq q_x, x \geq 1$ , then

$$\sum_{y=0}^{\infty} \gamma_y = 1 + \sum_{y=1}^{\infty} \frac{q_1 \cdots q_y}{p_1 \cdots p_y} \ge 1 + \sum_{y=1}^{\infty} 1^y = \infty.$$

Hence by Q26(a),  $\rho_{10} = 1$ . By one-step argument, we have

$$\rho_{00} = P(0,0)\rho_{00} + P(0,1)\rho_{10} = r_0\rho_{00} + p_0$$

Since  $p_0 + r_0 = 1$  and  $p_0 > 0$ , we have  $\rho_{00} = 1$ , that is, state 0 is recurrent. As the chain is irreducible, it is recurrent.

30 Solution. (a) Note that in Example 13,  $\gamma_x = 2(\frac{1}{x+1} - \frac{1}{x+2})$ . By (59), for a < x < b,

$$P_x(T_a < T_b) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y} = \frac{2(\frac{1}{x+1} - \frac{1}{b+1})}{2(\frac{1}{a+1} - \frac{1}{b+1})} = \frac{(a+1)(b-x)}{(x+1)(b-a)}.$$

(b) By Q26(b), for x > 0,

$$\rho_{x0} = \frac{\sum_{y=x}^{\infty} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y} = \frac{\frac{2}{x+1}}{2} = \frac{1}{x+1}.$$

31 If f(0) > 0, then for any x > 0,

$$P(x,0) = f(0)^x > 0.$$

Since 0 is absorbing, any positive x is transient.

If f(0) = 0, then  $X_n$  is nondecreasing, that is,  $\rho_{xy} = 0$  for x > y. Moreover, for x > 0,

$$\rho_{xx} = P(x, x) = f(1)^x < 1.$$

Hence any positive x is transient.

35 Note that for  $x \ge 1$ ,

$$\sum_{y} yP(x,y) = E_x(X_1) = E(\xi_1 + \xi_2 + \dots + \xi_x) = xE(\xi_1) = \mu x.$$

Using Q13(b), we have  $E_x(X_n) = \mu^n E_x(X_0) = x \mu^n$ .

36 Proof. (a)

$$E[X_{n+1}^2 \mid X_n = x] = E[(\xi_1 + \xi_2 + \dots + \xi_x)^2]$$
  
=  $\sum_{i=1}^x E(\xi_i^2) + 2 \sum_{1 \le i < j \le x} E(\xi_i \xi_j)$   
=  $\sum_{i=1}^x (E(\xi_i^2) - (E\xi_i)^2) + (\sum_{i=1}^x E\xi_i)^2$   
=  $x\sigma^2 + x^2\mu^2$ .

(b) Using Total Expectation Formula, Q36(a) and Q35, we have

$$E_x(X_{n+1}^2) = \sum_y P_x(X_n = y)E[X_{n+1}^2 | X_n = y]$$
  
=  $\sum_y P_x(X_n = y)(y\sigma^2 + y^2\mu^2)$   
=  $\sigma^2 \sum_y yP_x(X_n = y) + \mu^2 \sum_y y^2P_x(X_n = y)$   
=  $\sigma^2 E_x(X_n) + \mu^2 E_x(X_n^2)$   
=  $x\mu^n\sigma^2 + \mu^2 E_x(X_n^2).$ 

(c) Use induction on n. For n = 1, using Q36(a), we have

$$E_x(X_1^2) = x\sigma^2 + x^2\mu^2$$

Suppose that the formula holds for some  $n \ge 1$ , then

$$E_x(X_{n+1}^2) = x\mu^n \sigma^2 + \mu^2 E_x(X_n^2)$$
  
=  $x\mu^n \sigma^2 + \mu^2 (x\sigma^2(\mu^{n-1} + \dots + \mu^{2(n-1)}) + x^2\mu^{2n})$   
=  $x\sigma^2(\mu^n + \dots + \mu^{2n}) + x^2\mu^{2(n+1)}.$ 

Hence the formula also holds for n + 1.

(d) If there are x particles initially, then by Q35 and Q36(c), for  $n \ge 1$ ,

$$\operatorname{Var} X_n = E_x(X_n^2) - (E_x(X_n))^2 = \begin{cases} x\sigma^2\mu^{n-1}\left(\frac{1-\mu^n}{1-\mu}\right), & \mu \neq 1, \\ nx\sigma^2, & \mu = 1. \end{cases}$$

37 37. Proof. (a) If f(0) = 0, then P(x, x - 1) = f(0) = 0 for  $x \ge 1$ . That implies  $\rho_{xy} = 0$  for  $x > y \ge 0$ . Hence the chain is not irreducible.

If f(0) + f(1) = 1, then P(x, y) = f(y - x + 1) = 0 for  $1 \le x < y$ . That implies  $\rho_{xy} = 0$  for  $1 \le x < y$ . Hence the chain is not irreducible.

(b) For  $x > y \ge 0$ ,

$$\rho_{xy} \ge P(x, x-1)P(x-1, x-2) \cdots P(y+1, y) = (f(0))^{x-y} > 0$$

Since f(0) + f(1) < 1, there exists  $x_0 \ge 2$  such that  $f(x_0) > 0$ . Then for  $n \ge 0$ ,

$$\rho_{0,x_0+n(x_0-1)} \ge P(0,x_0)P(x_0,x_0+(x_0-1))P(x_0+(x_0-1),x_0+2(x_0-1))\cdots$$

$$P(x_0+(n-1)(x_0-1),x_0+n(x_0-1))$$

$$= f(x_0)^{n+1} > 0.$$

Now for any states x, y, there exists n such that  $x_0 + n(x_0 - 1) > y$ . Since x leads to 0, 0 leads to  $x_0 + n(x_0 - 1)$ ,  $x_0 + n(x_0 - 1)$  leads to y, x also leads to y. Hence the chain is irreducible.

38 (a) If f(1) = 1, all positive states 1, 2, ... are absorbing and recurrent, while 0 is transient.

(b) If f(0) > 0, f(1) > 0, and f(0) + f(1) = 1, states 0 and 1 are recurrent, while 2, 3, ... are transient.

- (c) If f(0) = 1, state 0 is absorbing and recurrent, while  $1, 2, \ldots$  are transient.
- (d) If f(0) = 0 and f(1) < 1, all states are transient.